ITERATIVE RECONSTRUCTION IN DIFFRACTION TOMOGRAPHY USING NON-UNIFORM FAST FOURIER TRANSFORM

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ABSTRACT

We show an iterative reconstruction framework for diffraction ultrasound tomography. The use of broadband illumination allows significantly reduce the number of projections compared to straight ray tomography. The proposed algorithm makes use of fast forward non-uniform Fourier transform (NUFFT) for iterative Fourier inversion. Incorporation of total variation regularization allows reduce noise and Gibbs phenomena whilst preserving the edges.

1. INTRODUCTION

Ultrasound tomography with diffracting sources is an important type of acoustic imaging. Since the used wave-lengths are comparable to the object feature dimensions, the straight ray tomography theory is no more applicable. Particularly, the Fourier Slice Theorem should be replaced by the Fourier Diffraction Theorem [11].

Image reconstruction in diffraction tomography can be considered as a problem of signal reconstruction from non-uniform frequency samples. Reconstruction methods used before addressed the problem as straightforward approximation of the inverse non-uniform Fourier transform (NUFT) and involved frequency interpolation [11], which is liable to introduce significant inaccuracies. More accurate and computationally efficient methods [2,6,7] were proposed for forward and inverse 1D NUFT. Fast forward NUFT algorithms can be generalized to higher dimensions, whereas the generalization of the inverse ones is not trivial. For this reason, we limit our work to the use of the forward NUFFT.

Recently, fast and accurate approximation of the forward NUFFT was introduced by Fessler and Sutton [8]. Inverse NUFFT can be achieved iteratively in this framework [9]. We adopt this approach for iterative reconstruction in diffraction tomography. We combine it with total variation regularization [4,10] in order to suppress noise preserving the sharpness of edges.

Simulation studies with the Shepp-Logan phantom show promising results.

2. PRINCIPLES OF DIFFRACTION TOMOGRAPHY

In diffraction ultrasound tomography, the object is illuminated with a plane acoustic wave. The forward scattered field is measured on a line of detectors, as shown in Figure 1. Data samples collected along this line are usually referred to as *projection*. As in straight ray tomography, changing the orientation of the incident plane waves, it is possible to acquire projections at different angles. Unlike conventional tomography, incident wave frequency can also be changed.

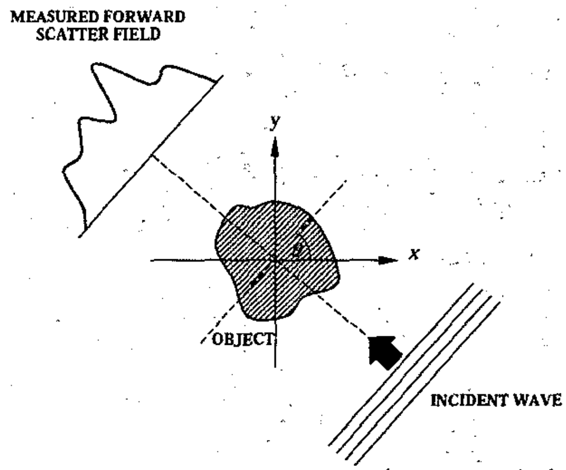

Fig. 1. Acquisition of a single projection in ultrasound diffraction tomography [11].

A fundamental tool of diffraction tomography is the Fourier Diffraction Theorem, which relates the Fourier transform of the measured scattered field projection with the Fourier transform of the object:

The Fourier Diffraction Theorem

Given a projection $P_\theta(r)$ of the forward scattered field obtained by illuminating an object $f(x)$ with a plane wave as shown in Figure 1, the following equation holds:

$$\mathcal{F}\{P_\theta(r)\}(\omega)=\mathcal{F}_{2D}\{f(x,y)\}(K_x(\omega),K_y(\omega))$$

where

$$K_x(\omega) = \omega\cos\theta - \left(\sqrt{K_0^2 - \omega^2} - K_0\right)\sin\theta,$$

$$K_y(\omega) = \omega\sin\theta + \left(\sqrt{K_0^2 - \omega^2} - K_0\right)\cos\theta,$$

$$K_0 = \frac{2\pi}{\lambda},$$

λ is the wavelength and $\mathcal{F}$ and $\mathcal{F}_{2D}$ denote the one- and two-dimensional Fourier transforms, respectively.

In other words, the Fourier transform of the projection gives the values of the 2D Fourier transform of the object along a semi-circular arc in the spatial frequency domain, as depicted in Figure 2. For proof see, for example, Kak and Slaney [11].

One can note that the arc radius becomes large as the wavelength shortens and the Diffraction Theorem approaches the Slice Theorem.

Since wave phenomena obey the superposition principle, illuminating the object with a wave consisting of a set of frequencies (referred to as *broadband illumination*), rather than a monochromatic wave, will produce samples along a set of semi-circular arcs with different radii. Hence, a single projection potentially contains much more information about the object than a single projection in straight ray tomography. By taking advantage of this fact, one can achieve sufficient image quality with few projections.

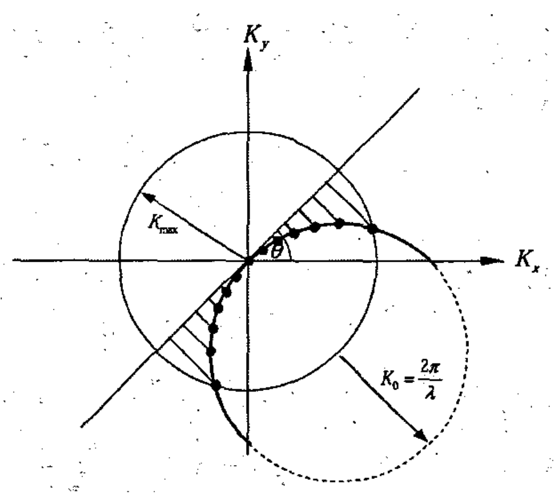

Figure 2. Illustration of the Fourier Diffraction Theorem.

3. THE NON-UNIFORM FOURIER TRANSFORM

The heart of iterative image reconstruction is the forward non-uniform fast Fourier transform (NUFFT). To define the NUFFT problem, we first consider a one-dimensional case. Let $\xi = (\xi_1, ..., \xi_N) : \xi_k \in [-\pi, \pi]$ be a vector of non-uniformly distributed frequencies and $x = (x_{-N/2}, ..., x_{N/2-1}) : x_n \in \mathbf{C}$ be a vector of samples of a signal. The non-uniform Fourier transform can be considered as an operator $\mathcal{T} : \mathbf{C}^N \to \mathbf{C}^K$, defined by the formula

$$X_k = (\mathcal{T}x)_k = \sum_{n=-N/2}^{N/2-1} x_n \exp(-ik\xi_k) \tag{1}$$

In matrix notation

$$X = \Psi x \tag{2}$$

where $\Psi \in \mathbf{C}^{K\times N}$ ($K \geq N$) is a full column rank matrix containing K discrete exponent functions in its rows

$$\Psi = (\psi_1; ...; \psi_K) \;\; ; \;\; \psi_k(n) = \exp(-in\xi_k) \tag{3}$$

Fast approximation of the NUFT operator can be achieved by projecting the signal x on some oversampled uniform Fourier basis $\Phi \in \mathbf{C}^{qN\times N}$ using standard FFT, with consequent efficient interpolation:

$$X \approx \mathrm{U}_p \Phi x \tag{4}$$

where U_p denotes the interpolation operator, which makes use of p neighboring uniform samples for approximation of each non-uniform sample. The overall complexity of such an algorithm is $O(qN\log qN + pK)$.

Fessler [8,9] proposed obtaining the interpolation coefficients from the solution of the following min-max problem:

$$\min_{u_k} \max_{x:\|x\|_2\leq 1} \left\| u_k \Phi_p^k x - \psi_k x \right\| \tag{5}$$

where u_k is the non-zero part of the k-th row of the interpolation matrix U_p and Φ_p^k is a part of the overcomplete DFT basis Φ, containing p nearest neighbors of ψ_k. The problem has an analytic solution:

$$u_k = \psi_k \Phi_p^{k^H} \left(\Phi_p^k \Phi_p^{k^H} \right)^{-1} \tag{6}$$

corresponding to the coefficients of the best approximation of ψ_k in Φ_p^k. In practice, the interpolation coefficients can be pre-computed.

4. ITERATIVE SOLUTION OF THE INVERSE PROBLEM

4.1. Formulation of the optimization problem

Straightforward solution of the inverse problem (2) is a computationally extensive operation. It is given by the Moore-Penrose pseudoinverse:

$$x = \Psi^+ X = \left(\Psi^{\mathrm{H}}\Psi\right)^{-1} \Psi^{\mathrm{H}} X \tag{7}$$

Alternatively, equation (2) can be reformulated as an optimization problem:

$$\min_{x} \|\mathcal{T}x - X\|_2^2 \qquad (8)$$

It is possible to add penalty on some kind of signal irregularity to the object function:

$$\min_{x} \|\mathcal{T}x - X\|_2^2 + \lambda\varphi(x) \qquad (9)$$

where λ is a parameter controlling the influence of the penalty. This problem can be solved iteratively using various optimization techniques, like Conjugate Gradients, Truncated Newton, etc. (see, for example [1]). They require efficient computation of the objective function and its gradient, which use in turn fast forward operator $\mathcal{T}$, and its adjoint $\mathcal{T}^{\dagger}$. The gradient of the cost function in (9) is given by

$$\nabla f(x) = 2\mathcal{T}^{\dagger}(\mathcal{T}x - X) + \nabla\varphi(x) \qquad (10)$$

4.2. Total variation regularization

Empirical observations show that the majority of images that occur in nature, and particularly in medical imaging applications, belong to the class of functions of bounded total variation (defined as the integral of the gradient l_2-norm) [12].

The penalty term for total variation can be used in problem (9). For discrete image, the total variation is given by

$$\sum_{i,j} \sqrt{(x_x)_{ij}^2 + (x_y)_{ij}^2} \qquad (11)$$

where x is the estimated discrete image being found during the iterative process and x_x, x_y are its discrete directional derivatives. Since this function is not smooth, which can be an obstacle for smooth optimization techniques, adding a positive smoothing parameter η, we finally get

$$\varphi(x) = \sum_{i,j} \sqrt{(x_x)_{ij}^2 + (x_y)_{ij}^2 + \eta} \qquad (12)$$

Total variation regularization removes small oscillations (resulting from noise and Gibbs phenomena), without significantly affecting the edges.

For analytic expression of the gradient and the Hessian of the cost function, see [3].

5. SIMULATIONS

In order to avoid forward-projection errors, we used an analytic Shepp-Logan phantom. This phantom is a superposition of ellipses representing features of the human brain. The advantage of such phantom is that its Fourier transform has a simple analytical expression.

Eight simulated broadband projections (each containing 10 frequencies) are shown in Figure 3. For comparison, in similar conditions, a conventional filtered backprojection (FBP) would require about 100 straight ray projections for good reconstruction results.

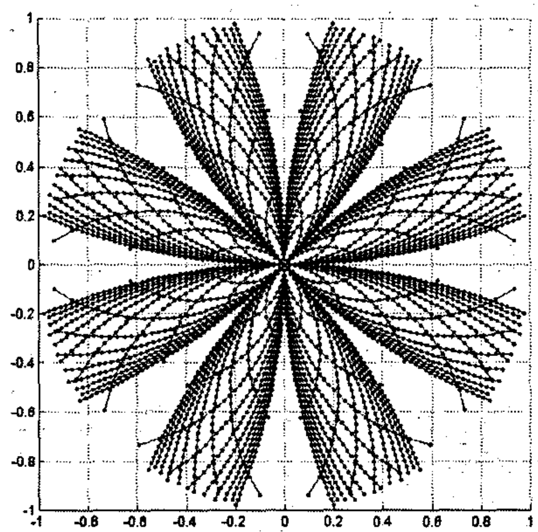

Figure 3. k-space sampling (spatial frequency is normalized to K_{max}).

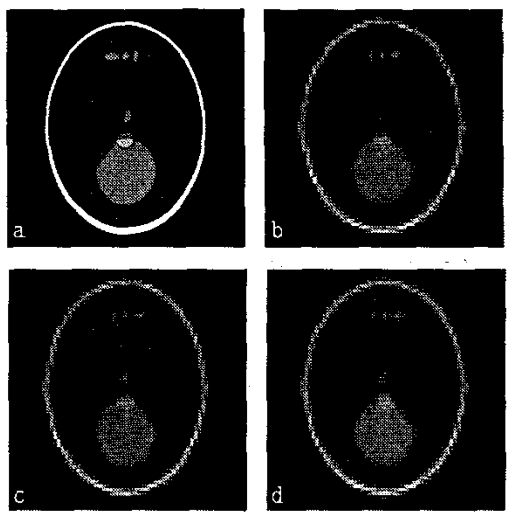

Figure 4. a. original band-unlimited phantom, b. iterative least-squares reconstruction, c. and d. iterative reconstruction with total variation penalty (λ = 10 and 100, respectively).

Fessler's NUFFT algorithm [8,9] was used as the forward operator. Images of size 64×64 were iteratively reconstructed using Conjugate Gradients (Figure 4).

Figure 5 shows images reconstructed from data contaminated by additive Gaussian noise with different variance. One can observe that the total variation penalty significantly improves the image quality.

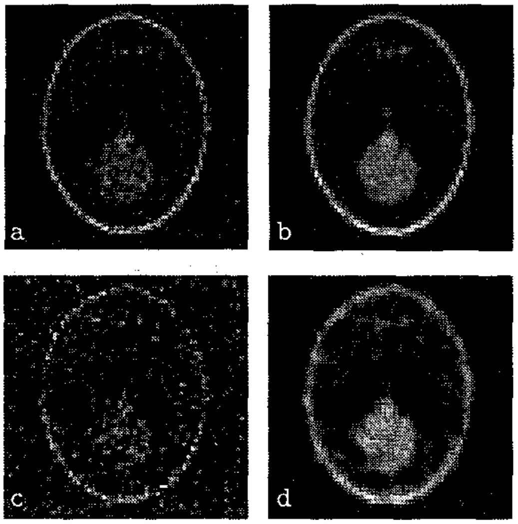

Figure 5. Iterative reconstruction in presence of noise (a, b: SNR = 20dB, c, d: SNR = 10dB) with total variation penalty $\lambda = 100$ (b, d) and without penalty (a, c).

6. CONCLUSIONS

In this work we showed an iterative reconstruction algorithm for ultrasound tomography with diffracting sources. The presented method is capable of taking advantage of broadband illumination and requires fewer projections for plausible image reconstruction.

Total variation regularization removes the Gibbs phenomena and suppresses the noise, thus allowing obtain sufficient reconstruction quality from noisy data when most methods fail to produce plausible results.

Possible developments of the iterative approach can be generalization of the regularizer for other classes of signals and incorporation of different NUFFT implementations. We intend studying novel non-smooth optimization techniques for efficient minimization of functions with non-smooth penalty terms.

7. ACKNOWLEDGEMENTS

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