## Graph Matching: Relax or Not?

## Alex Bronstein

School of Electrical Engineering
Tel Aviv University
College of Electrical and Computer Engineering
Duke University

NCSU, 2014

Joint work with Yonathan Aflalo and Ron Kimmel

## Minimum-distortion correspondences



## Minimum-distortion correspondences



Find the best structure-preserving correspondence

## Minimum-distortion correspondences



Find $\varphi:\left(X, d_{X}\right) \mapsto\left(Y, d_{Y}\right)$ minimizing $\left\|d_{X}-d_{Y} \circ(\varphi \times \varphi)\right\|$

## 'Graph matching' problems



## 'Graph matching' problems



## 'Graph matching' problems

Given two undirected weighted graphs represented by adjacency matrices $\mathbf{A}$ and $\mathbf{B}$

## 'Graph matching' problems

Given two undirected weighted graphs represented by adjacency matrices $\mathbf{A}$ and $\mathbf{B}$

Graph isomorphism: determine whether $\mathbf{A}$ and $\mathbf{B}$ are isomorphic

## 'Graph matching' problems

Given two undirected weighted graphs represented by adjacency matrices $\mathbf{A}$ and $\mathbf{B}$

Graph isomorphism: determine whether $\mathbf{A}$ and $\mathbf{B}$ are isomorphic

Exact graph 'matching': find isomorphism relating $\mathbf{A}$ and $\mathbf{B}$

## 'Graph matching' problems

Given two undirected weighted graphs represented by adjacency matrices $\mathbf{A}$ and $\mathbf{B}$

Graph isomorphism: determine whether $\mathbf{A}$ and $\mathbf{B}$ are isomorphic

Exact graph 'matching': find isomorphism relating $\mathbf{A}$ and $\mathbf{B}$

Inexact graph 'matching': find best approximate isomorphism relating $\mathbf{A}$ and $\mathbf{B}$

## Convex relaxation

## Graph Matching (NP)

$$
\boldsymbol{\Pi}^{*}=\underset{\Pi \in \mathcal{P}}{\operatorname{argmin}}\left\|\mathbf{A}-\boldsymbol{\Pi}^{\mathrm{T}} \mathbf{B} \boldsymbol{\Pi}\right\|
$$

$\mathcal{P}=$ space of $n \times n$ permutation matrices

## Convex relaxation

## Graph Matching (NP)

$$
\boldsymbol{\Pi}^{*}=\underset{\Pi \in \mathcal{P}}{\operatorname{argmin}}\left\|\mathbf{A}-\boldsymbol{\Pi}^{\mathrm{T}} \mathbf{B} \boldsymbol{\Pi}\right\|_{\mathrm{F}}^{2}
$$

$\mathcal{P}=$ space of $n \times n$ permutation matrices

## Convex relaxation

## Graph Matching (NP)

$$
\begin{gathered}
\boldsymbol{\Pi}^{*}=\underset{\boldsymbol{\Pi} \in \mathcal{P}}{\operatorname{argmin}}\|\boldsymbol{\Pi} \mathbf{A}-\mathbf{B \Pi}\|_{\mathrm{F}}^{2} \\
\mathcal{P}=\text { space of } n \times n \text { permutation matrices }
\end{gathered}
$$

## Convex relaxation

## Graph Matching (NP)

$$
\begin{gathered}
\mathbf{\Pi}^{*}=\underset{\Pi \in \mathcal{P}}{\operatorname{argmin}}\|\boldsymbol{\Pi} \mathbf{A}-\mathbf{B \Pi}\|_{\mathrm{F}}^{2} \\
\mathcal{P}=\text { space of } n \times n \text { permutation matrices }
\end{gathered}
$$

## Convex Relaxation

$$
\begin{gathered}
\mathbf{P}^{*}=\underset{\mathbf{P} \in \mathcal{D}}{\operatorname{argmin}}\|\mathbf{P} \mathbf{A}-\mathbf{B P}\|_{\mathrm{F}}^{2} \\
\mathcal{D}=\left\{\mathbf{P} \geq \mathbf{0}: \mathbf{P} \mathbf{1}=\mathbf{P}^{\mathrm{T}} \mathbf{1}=\mathbf{1}\right\} \text { space of } \\
n \times n \text { double-stochastic matrices }
\end{gathered}
$$

## Convex relaxation

## Graph Matching (NP)

$$
\begin{gathered}
\mathbf{\Pi}^{*}=\underset{\boldsymbol{\Pi} \in \mathcal{P}}{\operatorname{argmin}}\|\boldsymbol{\Pi} \mathbf{A}-\mathbf{B \Pi}\|_{\mathrm{F}}^{2} \\
\mathcal{P}=\text { space of } n \times n \text { permutation matrices }
\end{gathered}
$$

## Convex Relaxation (QP)

$$
\begin{gathered}
\mathbf{P}^{*}=\underset{\mathbf{P} \in \mathcal{D}}{\operatorname{argmin}}\|\mathbf{P} \mathbf{A}-\mathbf{B P}\|_{\mathrm{F}}^{2} \\
\mathcal{D}=\left\{\mathbf{P} \geq \mathbf{0}: \mathbf{P} \mathbf{1}=\mathbf{P}^{\mathrm{T}} \mathbf{1}=\mathbf{1}\right\} \text { space of } \\
n \times n \text { double-stochastic matrices }
\end{gathered}
$$

## Convex relaxation

## Convex Relaxation (QP)

$$
\mathbf{P}^{*}=\underset{\mathbf{P} \in \mathcal{D}}{\operatorname{argmin}}\|\mathbf{P} \mathbf{A}-\mathbf{B P}\|_{\mathrm{F}}^{2}
$$

Generally, $\mathbf{P}^{*}$ is not a permutation!

## Convex relaxation

## 1. Convex Relaxation (QP)

$$
\mathbf{P}^{*}=\underset{\mathbf{P} \in \mathcal{D}}{\operatorname{argmin}}\|\mathbf{P} \mathbf{A}-\mathbf{B P}\|_{\mathrm{F}}^{2}
$$

Generally, $\mathbf{P}^{*}$ is not a permutation!
2. Projection onto $\mathcal{P}$

$$
\hat{\boldsymbol{\Pi}}=\underset{\boldsymbol{\Pi} \in \mathcal{P}}{\operatorname{argmax}}\left\langle\boldsymbol{\Pi}, \mathbf{P}^{*}\right\rangle
$$

## Convex relaxation

## 1. Convex Relaxation (QP)

$$
\mathbf{P}^{*}=\underset{\mathbf{P} \in \mathcal{D}}{\operatorname{argmin}}\|\mathbf{P} \mathbf{A}-\mathbf{B P}\|_{\mathrm{F}}^{2}
$$

Generally, $\mathbf{P}^{*}$ is not a permutation!
2. Projection onto $\mathcal{P}$

$$
\hat{\boldsymbol{\Pi}}=\underset{\boldsymbol{\Pi} \in \mathcal{P}}{\operatorname{argmax}} \operatorname{tr}\left(\boldsymbol{\Pi}^{\mathrm{T}} \mathbf{P}^{*}\right)
$$

## Convex relaxation

## 1. Convex Relaxation (QP)

$$
\mathbf{P}^{*}=\underset{\mathbf{P} \in \mathcal{D}}{\operatorname{argmin}}\|\mathbf{P} \mathbf{A}-\mathbf{B P}\|_{\mathrm{F}}^{2}
$$

Generally, $\mathbf{P}^{*}$ is not a permutation!
2. Projection onto $\mathcal{P}$ (LAP)

$$
\hat{\boldsymbol{\Pi}}=\underset{\boldsymbol{\Pi} \in \mathcal{P}}{\operatorname{argmax}} \operatorname{tr}\left(\boldsymbol{\Pi}^{\mathrm{T}} \mathbf{P}^{*}\right)
$$

Solved by Hungarian algorithm

## Relax or not?

## What is the relation between $\Pi^{*}$ and $\hat{\Pi}$ ?

## Relax or not?

## What is the relation between $\Pi^{*}$ and $\hat{\Pi}$ ?

Obviously, $\boldsymbol{\Pi}^{*}$ is a solution of the relaxation

## Relax or not?

## What is the relation between $\Pi^{*}$ and $\hat{\Pi}$ ?

Obviously, $\Pi^{*}$ is a solution of the relaxation

However, the relaxation might produce some $\mathbf{P}^{*}$ which is not a permutation and its projection $\hat{\Pi}$ can have $\|\hat{\Pi} \mathbf{A}-\mathbf{B} \hat{\Pi}\|>0$

## Relax or not?

## What is the relation between $\Pi^{*}$ and $\hat{\Pi}$ ?

Obviously, $\boldsymbol{\Pi}^{*}$ is a solution of the relaxation

However, the relaxation might produce some $\mathbf{P}^{*}$ which is not a permutation and its projection $\hat{\Pi}$ can have $\|\hat{\Pi} \mathbf{A}-\mathbf{B} \hat{\Pi}\|>0$

Surprisingly, not so much is known about the relation between $\Pi^{*}$ and $\hat{\Pi}$ !

## Convex relaxation

## Convex Relaxation

$$
\begin{array}{r}
\mathbf{P}^{*}=\underset{\mathbf{P} \geq \mathbf{0}}{\operatorname{argmin}}\|\mathbf{P A}-\mathbf{B P}\|_{\mathrm{F}}^{2} \\
\text { s.t. } \mathbf{P} \mathbf{1}=\mathbf{P}^{\mathrm{T}} \mathbf{1}=\mathbf{1}
\end{array}
$$

double-stochastic matrices

## Convex relaxation

## An even bigger relaxation

$$
\begin{gathered}
\mathbf{P}^{*}=\underset{\mathbf{P}}{\operatorname{argmin}}\|\mathbf{P A}-\mathbf{B P}\|_{\mathrm{F}}^{2} \\
\text { s.t. } \mathbf{P} \mathbf{1}=\mathbf{1}
\end{gathered}
$$

pseduo-stochastic matrices

## Convex relaxation

## An even bigger relaxation

$$
\begin{gathered}
\mathbf{P}^{*}=\underset{\mathbf{P}}{\operatorname{argmin}}\|\mathbf{P A}-\mathbf{B P}\|_{\mathrm{F}}^{2} \\
\text { s.t. } \mathbf{P} \mathbf{1}=\mathbf{1}
\end{gathered}
$$

pseduo-stochastic matrices
$n$ non-overlapping equality constraints instead of $2 n$ overlapping constraints

## Convex relaxation

## An even bigger relaxation

$$
\begin{gathered}
\mathbf{P}^{*}=\underset{\mathbf{P}}{\operatorname{argmin}}\|\mathbf{P A}-\mathbf{B P}\|_{\mathrm{F}}^{2} \\
\text { s.t. } \mathbf{P} \mathbf{1}=\mathbf{1}
\end{gathered}
$$

pseduo-stochastic matrices
$n$ non-overlapping equality constraints instead of $2 n$ overlapping constraints
no inequality constraints

## Friendly graphs

## Convex Relaxation

$$
\mathbf{P}^{*}=\underset{\mathbf{P}}{\operatorname{argmin}}\|\mathbf{P} \mathbf{A}-\mathbf{B P}\|_{\mathrm{F}}^{2} \text { s.t. } \mathbf{P} \mathbf{1}=\mathbf{1}
$$

## Friendly graphs

## Convex Relaxation

$$
\mathbf{P}^{*}=\underset{\mathbf{P}}{\operatorname{argmin}}\|\mathbf{P} \mathbf{A}-\mathbf{B P}\|_{\mathrm{F}}^{2} \text { s.t. } \mathbf{P} \mathbf{1}=\mathbf{1}
$$

Friendly graphs: an undirected weighted graph A is friendly if

- A has simple spectrum
- no eigenvectors of A are orthogonal to the constant vector 1


## Friendly graphs

Property: friendly graphs are asymmetric

## Friendly graphs

Property: friendly graphs are asymmetric (have trivial automorphism group)

## Friendly graphs

Property: friendly graphs are asymmetric Proof: Let $\mathbf{A}=\mathbf{U} \Lambda \mathbf{U}^{\mathrm{T}}$ be friendly. Assume $\boldsymbol{\Pi} \neq \mathbf{I}$ permutation such that $\Pi \mathbf{A}=\mathbf{A} \boldsymbol{\Pi}$.

## Friendly graphs

Property: friendly graphs are asymmetric Proof: Let $\mathbf{A}=\mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^{\mathrm{T}}$ be friendly. Assume $\Pi \neq \mathbf{I}$ permutation such that $\Pi \mathbf{A}=\mathbf{A} \boldsymbol{\Pi}$.
$\Rightarrow \forall i: \mathbf{A u}_{i}=\lambda_{i} \mathbf{u}_{i}$

## Friendly graphs

Property: friendly graphs are asymmetric Proof: Let $\mathrm{A}=\mathbf{U} \boldsymbol{\Lambda} \mathrm{U}^{\mathrm{T}}$ be friendly. Assume $\boldsymbol{\Pi} \neq \mathbf{I}$ permutation such that $\Pi \mathbf{A}=\mathbf{A} \boldsymbol{\Pi}$.
$\Rightarrow \forall i: \Pi \mathbf{A u}_{i}=\lambda_{i} \boldsymbol{\Pi} \mathbf{u}_{i}$

## Friendly graphs

Property: friendly graphs are asymmetric Proof: Let $\mathbf{A}=\mathbf{U} \Lambda \mathbf{U}^{\mathrm{T}}$ be friendly. Assume $\boldsymbol{\Pi} \neq \mathbf{I}$ permutation such that $\Pi \mathbf{A}=\mathbf{A} \boldsymbol{\Pi}$.
$\Rightarrow \forall i: \quad \mathbf{A \Pi u}_{i}=\lambda_{i} \boldsymbol{\Pi} \mathbf{u}_{i}$

## Friendly graphs

Property: friendly graphs are asymmetric
Proof: Let $\mathbf{A}=\mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^{\mathrm{T}}$ be friendly.
Assume $\Pi \neq \mathbf{I}$ permutation such that $\Pi \mathbf{A}=\mathbf{A} \Pi$.
$\Rightarrow \forall i: \quad \mathbf{A} \boldsymbol{\Pi} \mathbf{u}_{i}=\lambda_{i} \boldsymbol{\Pi} \mathbf{u}_{i}$
$\Rightarrow \Pi \mathbf{u}_{i}$ is an eigenvector of $\mathbf{A}$ corresponding to $\lambda_{i}$.

## Friendly graphs

Property: friendly graphs are asymmetric
Proof: Let $\mathbf{A}=\mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^{\mathrm{T}}$ be friendly.
Assume $\Pi \neq \mathbf{I}$ permutation such that $\Pi \mathbf{A}=\mathbf{A} \Pi$.
$\Rightarrow \forall i: \quad \mathbf{A} \Pi \mathbf{u}_{i}=\lambda_{i} \boldsymbol{\Pi} \mathbf{u}_{i}$
$\Rightarrow \Pi \mathbf{u}_{i}$ is an eigenvector of $\mathbf{A}$ corresponding to $\lambda_{i}$.
A has simple spectrum $\Rightarrow \Pi \mathbf{u}_{i}= \pm \mathbf{u}_{i}$.

## Friendly graphs

Property: friendly graphs are asymmetric Proof: Let $\mathbf{A}=\mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^{\mathrm{T}}$ be friendly. Assume $\boldsymbol{\Pi} \neq \mathbf{I}$ permutation such that $\Pi \mathbf{A}=\mathbf{A} \boldsymbol{\Pi}$.
$\Rightarrow \forall i: \quad \mathbf{A \Pi u}_{i}=\lambda_{i} \boldsymbol{\Pi u}_{i}$
$\Rightarrow \Pi \mathbf{u}_{i}$ is an eigenvector of $\mathbf{A}$ corresponding to $\lambda_{i}$.
A has simple spectrum $\Rightarrow \Pi \mathbf{u}_{i}= \pm \mathbf{u}_{i}$.
$\Pi \neq \mathbf{I} \Rightarrow \exists \mathbf{u}_{i}$ for which $\Pi \mathbf{u}_{i}=-\mathbf{u}_{i}$

## Friendly graphs

Property: friendly graphs are asymmetric Proof: Let $\mathbf{A}=\mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^{\mathrm{T}}$ be friendly. Assume $\boldsymbol{\Pi} \neq \mathbf{I}$ permutation such that $\Pi \mathbf{A}=\mathbf{A} \boldsymbol{\Pi}$.
$\Rightarrow \forall i: \quad \mathbf{A} \Pi \mathbf{u}_{i}=\lambda_{i} \boldsymbol{\Pi} \mathbf{u}_{i}$
$\Rightarrow \Pi \mathbf{u}_{i}$ is an eigenvector of $\mathbf{A}$ corresponding to $\lambda_{i}$.
A has simple spectrum $\Rightarrow \Pi \mathbf{u}_{i}= \pm \mathbf{u}_{i}$.
$\boldsymbol{\Pi} \neq \mathbf{I} \Rightarrow \exists \mathbf{u}_{i}$ for which $\Pi \mathbf{u}_{i}=-\mathbf{u}_{i}$
$\Rightarrow \mathbf{1}^{\mathrm{T}} \Pi \mathbf{u}_{i}=-\mathbf{1}^{\mathrm{T}} \mathbf{u}_{i}$.

## Friendly graphs

Property: friendly graphs are asymmetric Proof: Let $\mathbf{A}=\mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^{\mathrm{T}}$ be friendly.
Assume $\boldsymbol{\Pi} \neq \mathbf{I}$ permutation such that $\Pi \mathbf{A}=\mathbf{A} \boldsymbol{\Pi}$.
$\Rightarrow \forall i: \quad \mathbf{A} \Pi \mathbf{u}_{i}=\lambda_{i} \boldsymbol{\Pi} \mathbf{u}_{i}$
$\Rightarrow \Pi \mathbf{u}_{i}$ is an eigenvector of $\mathbf{A}$ corresponding to $\lambda_{i}$.
A has simple spectrum $\Rightarrow \Pi \mathbf{u}_{i}= \pm \mathbf{u}_{i}$.
$\boldsymbol{\Pi} \neq \mathbf{I} \Rightarrow \exists \mathbf{u}_{i}$ for which $\Pi \mathbf{u}_{i}=-\mathbf{u}_{i}$
$\Rightarrow \mathbf{1}^{\mathrm{T}} \boldsymbol{\Pi u}_{i}=-\mathbf{1}^{\mathrm{T}} \mathbf{u}_{i}$.
$\Pi$ is a permutation $\Rightarrow \mathbf{1}^{\mathrm{T}} \boldsymbol{\Pi}=\mathbf{1}^{\mathrm{T}}$

## Friendly graphs

Property: friendly graphs are asymmetric Proof: Let $\mathbf{A}=\mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^{\mathrm{T}}$ be friendly.
Assume $\boldsymbol{\Pi} \neq \mathbf{I}$ permutation such that $\Pi \mathbf{A}=\mathbf{A} \boldsymbol{\Pi}$.
$\Rightarrow \forall i: \quad \mathbf{A} \Pi \mathbf{u}_{i}=\lambda_{i} \boldsymbol{\Pi} \mathbf{u}_{i}$
$\Rightarrow \Pi \mathbf{u}_{i}$ is an eigenvector of $\mathbf{A}$ corresponding to $\lambda_{i}$.
A has simple spectrum $\Rightarrow \Pi \mathbf{u}_{i}= \pm \mathbf{u}_{i}$.
$\Pi \neq \mathbf{I} \Rightarrow \exists \mathbf{u}_{i}$ for which $\Pi \mathbf{u}_{i}=-\mathbf{u}_{i}$
$\Rightarrow \mathbf{1}^{\mathrm{T}} \boldsymbol{\Pi u}_{i}=-\mathbf{1}^{\mathrm{T}} \mathbf{u}_{i}$.
$\Pi$ is a permutation $\Rightarrow \mathbf{1}^{\mathrm{T}} \Pi \mathbf{u}_{i}=\mathbf{1}^{\mathrm{T}} \mathbf{u}_{i}$

## Friendly graphs

Property: friendly graphs are asymmetric Proof: Let $\mathbf{A}=\mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^{\mathrm{T}}$ be friendly.
Assume $\boldsymbol{\Pi} \neq \mathbf{I}$ permutation such that $\Pi \mathbf{A}=\mathbf{A} \boldsymbol{\Pi}$.
$\Rightarrow \forall i: \mathbf{A \Pi u}_{i}=\lambda_{i} \boldsymbol{\Pi} \mathbf{u}_{i}$
$\Rightarrow \Pi \mathbf{u}_{i}$ is an eigenvector of A corresponding to $\lambda_{i}$.
A has simple spectrum $\Rightarrow \Pi \mathbf{u}_{i}= \pm \mathbf{u}_{i}$.
$\Pi \neq \mathbf{I} \Rightarrow \exists \mathbf{u}_{i}$ for which $\Pi \mathbf{u}_{i}=-\mathbf{u}_{i}$
$\Rightarrow \mathbf{1}^{\mathrm{T}} \boldsymbol{\Pi u}_{i}=-\mathbf{1}^{\mathrm{T}} \mathbf{u}_{i}$.
$\Pi$ is a permutation $\Rightarrow 1^{\mathrm{T}} \Pi \mathbf{u}_{i}=\mathbf{1}^{\mathrm{T}} \mathbf{u}_{i}$
$\Rightarrow \mathbf{1}^{\mathrm{T}} \mathbf{u}_{i}=0$ in contradiction to friendliness

## Friendly graphs

Property: friendly graphs are asymmetric Proof: Let $\mathbf{A}=\mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^{\mathrm{T}}$ be friendly.
Assume $\Pi \neq \mathbf{I}$ permutation such that $\Pi \mathbf{A}=\mathbf{A} \Pi$.
$\Rightarrow \forall i: \quad \mathbf{A} \Pi \mathbf{u}_{i}=\lambda_{i} \boldsymbol{\Pi} \mathbf{u}_{i}$
$\Rightarrow \Pi \mathbf{u}_{i}$ is an eigenvector of $\mathbf{A}$ corresponding to $\lambda_{i}$.
A has simple spectrum $\Rightarrow \Pi \mathbf{u}_{i}= \pm \mathbf{u}_{i}$.
$\boldsymbol{\Pi} \neq \mathbf{I} \Rightarrow \exists \mathbf{u}_{i}$ for which $\boldsymbol{\Pi} \mathbf{u}_{i}=-\mathbf{u}_{i}$
$\Rightarrow \mathbf{1}^{\mathrm{T}} \boldsymbol{\Pi} \mathbf{u}_{i}=-\mathbf{1}^{\mathrm{T}} \mathbf{u}_{i}$.
$\Pi$ is a permutation $\Rightarrow 1^{\mathrm{T}} \boldsymbol{\Pi} \mathbf{u}_{i}=\mathbf{1}^{\mathrm{T}} \mathbf{u}_{i}$
$\Rightarrow \mathbf{1}^{\mathrm{T}} \mathbf{u}_{i}=0$ in contradiction to friendliness
Converse is not true (think of a regular asymmetric graph), but such graphs should be rare

## Main result

Theorem: Let $\mathbf{A}$ and $\mathbf{B}$ be friendly isomorphic graphs. Then $\mathbf{P}^{*}=\boldsymbol{\Pi}^{*}$.

## Main result

Theorem: Let $\mathbf{A}$ and $\mathbf{B}$ be friendly isomorphic graphs. Then $\hat{\Pi}=\mathbf{P}^{*}=\Pi^{*}$.

## Main result

Theorem: Let $\mathbf{A}$ and $\mathbf{B}$ be friendly isomorphic graphs. Then $\hat{\Pi}=\mathbf{P}^{*}=\Pi^{*}$.

Checking isomorphism is hard

## Main result

Theorem: Let $\mathbf{A}$ and $\mathbf{B}$ be friendly isomorphic graphs. Then $\hat{\Pi}=\mathbf{P}^{*}=\Pi^{*}$.

Checking isomorphism is hard
Checking friendliness is easy

## Main result

Theorem: Let A and $\mathbf{B}$ be friendly isomorphic graphs. Then $\hat{\Pi}=\mathbf{P}^{*}=\Pi^{*}$.

Checking isomorphism is hard
Checking friendliness is easy
Solve the relaxation: if $\mathbf{P}^{*} \mathbf{A}=\mathbf{B P} \mathbf{P}^{*}$ then the unique isomorphism is $\Pi^{*}=\mathbf{P}^{*}$.
Otherwise, no isomorphism exists.

## Sketch of the proof

Input: two friendly graphs $\mathbf{B}$ and $\mathbf{A}=\Pi^{* T} \mathbf{B} \Pi^{*}$

## Sketch of the proof

Input: two friendly graphs $\mathbf{B}$ and $\mathbf{A}=\boldsymbol{\Pi}^{* T} \mathbf{B} \Pi^{*}$
Convex quadratic program

$$
\min _{\mathbf{P}}\|\mathbf{P A}-\mathbf{B P}\|_{F}^{2} \text { s.t. } \mathbf{P} \mathbf{1}=\mathbf{1}
$$

with global minimizer $\mathbf{P}=\boldsymbol{\Pi}^{*}$.

## Sketch of the proof

Input: two friendly graphs $\mathbf{B}$ and $\mathbf{A}=\boldsymbol{\Pi}^{* T} \mathbf{B} \Pi^{*}$
Convex quadratic program

$$
\min _{\mathbf{P}}\|\mathbf{P A}-\mathbf{B P}\|_{F}^{2} \text { s.t. } \mathbf{P} \mathbf{1}=\mathbf{1}
$$

with global minimizer $\mathbf{P}=\boldsymbol{\Pi}^{*}$.
Show that the minimizer is unique

## Sketch of the proof

Input: two friendly graphs $\mathbf{B}$ and $\mathbf{A}=\boldsymbol{\Pi}^{* T} \mathbf{B} \Pi^{*}$
Convex quadratic program

$$
\min _{\mathbf{P}}\left\|\mathbf{P} \Pi^{* T} \mathbf{B} \Pi^{*}-\mathbf{B} \mathbf{P}\right\|_{F}^{2} \text { s.t. } \mathbf{P} \mathbf{1}=\mathbf{1}
$$

with global minimizer $\mathbf{P}=\boldsymbol{\Pi}^{*}$.
Show that the minimizer is unique

## Sketch of the proof

Input: two friendly graphs $\mathbf{B}$ and $\mathbf{A}=\boldsymbol{\Pi}^{* T} \mathbf{B} \Pi^{*}$
Convex quadratic program

$$
\min _{\mathbf{P}}\left\|\mathbf{P} \Pi^{* \mathrm{~T}} \mathbf{B}-\mathbf{B} \mathbf{P} \Pi^{* \mathrm{~T}}\right\|_{\mathrm{F}}^{2} \text { s.t. } \mathbf{P} \mathbf{1}=\mathbf{1}
$$

with global minimizer $\mathbf{P}=\boldsymbol{\Pi}^{*}$.
Show that the minimizer is unique

## Sketch of the proof

Input: two friendly graphs $\mathbf{B}$ and $\mathbf{A}=\boldsymbol{\Pi}^{* T} \mathbf{B} \Pi^{*}$
Convex quadratic program

$$
\min _{\mathbf{P}}\left\|\mathbf{P} \Pi^{* \mathrm{~T}} \mathbf{B}-\mathbf{B P} \Pi^{* \mathrm{~T}}\right\|_{\mathrm{F}}^{2} \text { s.t. } \mathbf{P} \boldsymbol{\Pi}^{* \mathrm{~T}} \mathbf{1}=\mathbf{1}
$$

with global minimizer $\mathbf{P}=\boldsymbol{\Pi}^{*}$.
Show that the minimizer is unique

## Sketch of the proof

Input: two friendly graphs $\mathbf{B}$ and $\mathbf{A}=\Pi^{* T} \mathbf{B} \boldsymbol{\Pi}^{*}$
Convex quadratic program reparametrized with $\mathbf{Q}=\mathbf{P} \Pi^{* T}$

$$
\min _{\mathrm{Q}}\|\mathrm{QB}-\mathrm{BQ}\|_{\mathrm{F}}^{2} \text { s.t. } \mathrm{Q} 1=\mathbf{1}
$$

with global minimizer $\mathrm{Q}=\boldsymbol{\Pi}^{*} \Pi^{* T}=\mathbf{I}$.
Show that the minimizer is unique

## Sketch of the proof

## $\min _{\mathbf{Q}}\|\mathrm{QB}-\mathrm{BQ}\|_{\mathrm{F}}^{2}$ s.t. $\mathrm{Q} 1=1$

## Sketch of the proof

$$
\min _{\mathbf{Q}}\|\mathbf{Q B}-\mathbf{B Q}\|_{\mathrm{F}}^{2} \text { s.t. } \mathrm{Q} 1=1
$$

First-order optimality condition: There exit $n$ Lagrange multipliers $\boldsymbol{\alpha}$ such that

$$
\mathbf{0}=\nabla_{\mathbf{Q}} \mathcal{L}=\mathbf{Q B}^{2}+\mathbf{B}^{2} \mathbf{Q}-2 \mathbf{B Q B}+\boldsymbol{\alpha} \mathbf{1}^{\mathrm{T}}
$$

## Sketch of the proof

$$
\min _{\mathbf{Q}}\|\mathbf{Q B}-\mathbf{B Q}\|_{\mathrm{F}}^{2} \text { s.t. } \mathrm{Q} 1=\mathbf{1}
$$

First-order optimality condition: using spectral representation $\mathbf{B}=\mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^{\mathrm{T}}$

$$
\mathbf{0}=\nabla_{\mathbf{Q}} \mathcal{L}=\mathbf{Q B}^{2}+\mathbf{B}^{2} \mathbf{Q}-2 \mathbf{B Q B}+\boldsymbol{\alpha} \mathbf{1}^{\mathrm{T}}
$$

## Sketch of the proof

$$
\min _{\mathbf{Q}}\|\mathbf{Q B}-\mathbf{B Q}\|_{\mathrm{F}}^{2} \text { s.t. } \mathrm{Q} 1=\mathbf{1}
$$

First-order optimality condition: using spectral representation $\mathbf{B}=\mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^{\mathrm{T}}$

$$
\begin{aligned}
\mathbf{0}= & \mathbf{Q U} \boldsymbol{\Lambda}^{2} \mathbf{U}^{\mathrm{T}}+\mathbf{U} \boldsymbol{\Lambda}^{2} \mathbf{U}^{\mathrm{T}} \mathbf{Q} \\
& -2 \mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^{\mathrm{T}} \mathbf{Q U} \boldsymbol{\Lambda} \mathbf{U}^{\mathrm{T}}+\boldsymbol{\alpha} \mathbf{1}^{\mathrm{T}}
\end{aligned}
$$

## Sketch of the proof

$$
\min _{\mathbf{Q}}\|\mathbf{Q B}-\mathbf{B Q}\|_{\mathrm{F}}^{2} \text { s.t. } \mathrm{Q} 1=1
$$

First-order optimality condition: using spectral representation $\mathbf{B}=\mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^{\mathrm{T}}$

$$
\begin{aligned}
\mathbf{0}= & \mathrm{U}^{\mathrm{T}} \mathbf{Q U} \boldsymbol{\Lambda}^{2}+\boldsymbol{\Lambda}^{2} \mathbf{U}^{\mathrm{T}} \mathbf{Q U} \\
& -2 \boldsymbol{\Lambda} \mathbf{U}^{\mathrm{T}} \mathbf{Q U} \boldsymbol{\Lambda}+\mathbf{U}^{\mathrm{T}} \boldsymbol{\alpha} \mathbf{1}^{\mathrm{T}} \mathbf{U}
\end{aligned}
$$

## Sketch of the proof

$$
\min _{\mathbf{Q}}\|\mathbf{Q B}-\mathbf{B Q}\|_{\mathrm{F}}^{2} \text { s.t. } \mathrm{Q} 1=1
$$

First-order optimality condition: using spectral representation $\mathbf{B}=\mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^{\mathrm{T}}$

$$
\mathbf{0}=\mathbf{F} \boldsymbol{\Lambda}^{2}+\boldsymbol{\Lambda}^{2} \mathbf{F}-2 \boldsymbol{\Lambda} \mathbf{F} \boldsymbol{\Lambda}+\gamma \mathbf{v}^{\mathrm{T}}
$$

where $\mathbf{F}=\mathbf{U}^{\mathrm{T}} \mathbf{Q U}, \boldsymbol{\gamma}=\mathrm{U}^{\mathrm{T}} \boldsymbol{\alpha}, \mathbf{v}=\mathrm{U}^{\mathrm{T}} \mathbf{1}$

## Sketch of the proof

First-order optimality condition:

$$
\mathbf{F} \boldsymbol{\Lambda}^{2}+\boldsymbol{\Lambda}^{2} \mathbf{F}-2 \boldsymbol{\Lambda} \mathbf{F} \boldsymbol{\Lambda}+\gamma \mathbf{v}^{\mathrm{T}}=\mathbf{0}
$$

## Sketch of the proof

First-order optimality condition:

$$
F_{i j}\left(\lambda_{i}-\lambda_{j}\right)^{2}+v_{j} \gamma_{i}=0
$$

## Sketch of the proof

## First-order optimality condition:

$$
F_{i j}\left(\lambda_{i}-\lambda_{j}\right)^{2}+v_{j} \gamma_{i}=0
$$

In particular, for $i=j: v_{i} \gamma_{i}=0$

## Sketch of the proof

## First-order optimality condition:

$$
F_{i j}\left(\lambda_{i}-\lambda_{j}\right)^{2}+v_{j} \gamma_{i}=0
$$

In particular, for $i=j: v_{i} \gamma_{i}=0$
Due to friendliness $v_{i}=\mathbf{u}_{i}^{\mathrm{T}} \mathbf{1} \neq 0$

## Sketch of the proof

## First-order optimality condition:

$$
F_{i j}\left(\lambda_{i}-\lambda_{j}\right)^{2}+v_{j} \gamma_{i}=0
$$

In particular, for $i=j: v_{i} \gamma_{i}=0$
Due to friendliness $v_{i}=\mathbf{u}_{i}^{\mathrm{T}} \mathbf{1} \neq 0 \Rightarrow \gamma=\mathbf{0}$

## Sketch of the proof

## First-order optimality condition:

$$
F_{i j}\left(\lambda_{i}-\lambda_{j}\right)^{2}=0 \quad \text { for } i \neq j
$$

## Sketch of the proof

## First-order optimality condition:

$$
F_{i j}\left(\lambda_{i}-\lambda_{j}\right)^{2}=0 \quad \text { for } i \neq j
$$

Due to friendliness $\lambda_{i} \neq \lambda_{j}$

## Sketch of the proof

## First-order optimality condition:

$$
F_{i j}\left(\lambda_{i}-\lambda_{j}\right)^{2}=0 \quad \text { for } i \neq j
$$

Due to friendliness $\lambda_{i} \neq \lambda_{j} \Rightarrow \mathbf{F}$ is diagonal

## Sketch of the proof

## First-order optimality condition:

$$
F_{i j}\left(\lambda_{i}-\lambda_{j}\right)^{2}=0 \quad \text { for } i \neq j
$$

Due to friendliness $\lambda_{i} \neq \lambda_{j} \Rightarrow \mathbf{F}$ is diagonal

$$
1=\mathrm{Q} 1
$$

## Sketch of the proof

## First-order optimality condition:

$$
F_{i j}\left(\lambda_{i}-\lambda_{j}\right)^{2}=0 \quad \text { for } i \neq j
$$

Due to friendliness $\lambda_{i} \neq \lambda_{j} \Rightarrow \mathbf{F}$ is diagonal

$$
\mathbf{1}=\mathrm{Q} \mathbf{1}=\mathrm{UFU}^{\mathrm{T}} \mathbf{1}
$$

## Sketch of the proof

## First-order optimality condition:

$$
F_{i j}\left(\lambda_{i}-\lambda_{j}\right)^{2}=0 \quad \text { for } i \neq j
$$

Due to friendliness $\lambda_{i} \neq \lambda_{j} \Rightarrow \mathbf{F}$ is diagonal

$$
\mathbf{1}=\mathrm{Q} \mathbf{1}=\mathrm{UFU}^{\mathrm{T}} \mathbf{1} \Rightarrow \mathrm{U}^{\mathrm{T}} \mathbf{1}=\mathbf{F U}^{\mathrm{T}} \mathbf{1}
$$

## Sketch of the proof

## First-order optimality condition:

$$
F_{i j}\left(\lambda_{i}-\lambda_{j}\right)^{2}=0 \quad \text { for } i \neq j
$$

Due to friendliness $\lambda_{i} \neq \lambda_{j} \Rightarrow \mathbf{F}$ is diagonal

$$
\mathbf{1}=\mathbf{Q} \mathbf{1}=\mathrm{UFU}^{\mathrm{T}} \mathbf{1} \Rightarrow \mathbf{U}^{\mathrm{T}} \mathbf{1}=\mathbf{F U}^{\mathrm{T}} \mathbf{1}
$$

$$
\Rightarrow \mathbf{v}=\mathbf{F} \mathbf{v} \text { with } v_{i} \neq 0
$$

## Sketch of the proof

## First-order optimality condition:

$$
F_{i j}\left(\lambda_{i}-\lambda_{j}\right)^{2}=0 \quad \text { for } i \neq j
$$

Due to friendliness $\lambda_{i} \neq \lambda_{j} \Rightarrow \mathbf{F}$ is diagonal

$$
\mathbf{1}=\mathbf{Q} \mathbf{1}=\mathrm{UFU}^{\mathrm{T}} \mathbf{1} \Rightarrow \mathbf{U}^{\mathrm{T}} \mathbf{1}=\mathbf{F U}^{\mathrm{T}} \mathbf{1}
$$

$$
\Rightarrow \mathbf{v}=\mathbf{F} \mathbf{v} \text { with } v_{i} \neq 0 \Rightarrow \mathbf{F}=\mathbf{I}
$$

## Sketch of the proof

## First-order optimality condition:

$$
F_{i j}\left(\lambda_{i}-\lambda_{j}\right)^{2}=0 \quad \text { for } i \neq j
$$

Due to friendliness $\lambda_{i} \neq \lambda_{j} \Rightarrow \mathbf{F}$ is diagonal
$\mathbf{1}=\mathbf{Q} \mathbf{1}=\mathbf{U F U}^{\mathrm{T}} \mathbf{1} \Rightarrow \mathbf{U}^{\mathrm{T}} \mathbf{1}=\mathbf{F U}^{\mathrm{T}} \mathbf{1}$
$\Rightarrow \mathbf{v}=\mathbf{F} \mathbf{v}$ with $v_{i} \neq 0 \Rightarrow \mathbf{F}=\mathbf{I}$
$\Rightarrow \mathbf{Q}=\mathbf{U F U}^{\mathrm{T}}=\mathbf{I}$

## Inexact graph matching

## Friendliness:

- A has simple spectrum
- no eigenvectors of $\mathbf{A}$ are orthogonal to the constant vector 1

Theorem: Let $\mathbf{A}$ and $\mathbf{B}$ be friendly isomorphic graphs. Then $\hat{\Pi}=\mathbf{P}^{*}=\mathbf{\Pi}^{*}$.

## Inexact graph matching

## Strong friendliness:

- A has $\delta$-separated spectrum
- every eigenvector $\mathbf{u}_{i}$ of $\mathbf{A}$ satisfied $\left|\mathbf{u}_{i}^{\mathrm{T}} \mathbf{1}\right|>\epsilon$

Theorem: Let $\mathbf{A}$ and $\mathbf{B}$ be strongly friendly $\rho$-isomorphic graphs with $\rho=\rho(\epsilon, \delta)$. Then $\left\|\mathbf{P}^{*}-\boldsymbol{\Pi}^{*}\right\|_{\infty}<\frac{1}{2}$.
$\rho$-isomorphic $\Leftrightarrow \exists \boldsymbol{\Pi}^{*}:\left\|\boldsymbol{\Pi}^{*} \mathbf{A}-\mathbf{B} \boldsymbol{\Pi}^{*}\right\|_{\mathrm{F}}^{2} \leq \rho$

## Inexact graph matching

## Strong friendliness:

- A has $\delta$-separated spectrum
- every eigenvector $\mathbf{u}_{i}$ of $\mathbf{A}$ satisfied $\left|\mathbf{u}_{i}^{\mathrm{T}} \mathbf{1}\right|>\epsilon$

Theorem: Let $\mathbf{A}$ and $\mathbf{B}$ be strongly friendly $\rho$-isomorphic graphs with $\rho=\rho(\epsilon, \delta)$. Then $\left\|\mathbf{P}^{*}-\boldsymbol{\Pi}^{*}\right\|_{\infty}<\frac{1}{2}$.

Proof using results from regular perturbation theory of linear equations

## Inexact graph matching

## Strong friendliness:

- A has $\delta$-separated spectrum
- every eigenvector $\mathbf{u}_{i}$ of $\mathbf{A}$ satisfied $\left|\mathbf{u}_{i}^{\mathrm{T}} \mathbf{1}\right|>\epsilon$

Theorem: Let $\mathbf{A}$ and $\mathbf{B}$ be strongly friendly $\rho$-isomorphic graphs with $\rho=\rho(\epsilon, \delta)$. Then
$\hat{\boldsymbol{\Pi}}=\boldsymbol{\Pi}^{*}$.
Proof using results from regular perturbation theory of linear equations

## Inexact graph matching

## Strong friendliness:

- A has $\delta$-separated spectrum
- every eigenvector $\mathbf{u}_{i}$ of $\mathbf{A}$ satisfied $\left|\mathbf{u}_{i}^{\mathrm{T}} \mathbf{1}\right|>\epsilon$

Theorem: Let $\mathbf{A}$ and $\mathbf{B}$ be strongly friendly $\rho$-isomorphic graphs with $\rho=\rho(\epsilon, \delta)$. Then
$\hat{\boldsymbol{\Pi}}=\boldsymbol{\Pi}^{*}$.
If $\left\|\mathbf{P}^{*} \mathbf{A}-\mathbf{B P}^{*}\right\|_{\mathrm{F}}^{2}<\rho(\epsilon, \delta)$ then $\hat{\boldsymbol{\Pi}}$ is the globally optimal approximate isomorphism. Otherwise, no $\rho$-isomorphism exists.

## Experimental validation on 1000 strongly friendly graphs



## Unfriendly graphs

Adjacency matrix has $d$ non-simple eigenspaces

$$
\underbrace{\lambda_{1}=\lambda_{2}=\cdots=\lambda_{i_{1}}}_{\text {multiplicity } m_{1}+1}<\underbrace{\lambda_{i_{1}+1}=\cdots=\lambda_{i_{1}+i_{2}}}_{\text {multiplicity } m_{2}+1}<\cdots
$$

## Unfriendly graphs

Adjacency matrix has $d$ non-simple eigenspaces

$$
\begin{aligned}
& \underbrace{\lambda_{1}=\lambda_{2}=\cdots=\lambda_{i_{1}}}_{\text {multiplicity } m_{1}+1}<\underbrace{\lambda_{i_{1}+1}=\cdots=\lambda_{i_{1}+i_{2}}}_{\text {multiplicity } m_{2}+1}<\cdots \\
& m=m_{1}+m_{2}+\cdots+m_{d}
\end{aligned}
$$

## Unfriendly graphs

Adjacency matrix has $d$ non-simple eigenspaces

$$
\underbrace{\lambda_{1}=\lambda_{2}=\cdots=\lambda_{i_{1}}}_{\text {multiplicity } m_{1}+1}<\underbrace{\lambda_{i_{1}+1}=\cdots=\lambda_{i_{1}+i_{2}}}_{\text {multiplicity } m_{2}+1}<\cdots
$$

$$
m=m_{1}+m_{2}+\cdots+m_{d}
$$

Basis vectors of each eigenspace are selected such that either
none of them is orthogonal to 1 ; or all are orthogonal to 1

## Unfriendly graphs

Adjacency matrix has $d$ non-simple eigenspaces

$$
\underbrace{\lambda_{1}=\lambda_{2}=\cdots=\lambda_{i_{1}}}_{\text {multiplicity } m_{1}+1}<\underbrace{\lambda_{i_{1}+1}=\cdots=\lambda_{i_{1}+i_{2}}}_{\text {multiplicity } m_{2}+1}<\cdots
$$

$$
m=m_{1}+m_{2}+\cdots+m_{d}
$$

Basis vectors of each eigenspace are selected such that either
none of them is orthogonal to 1 (non-hostile); or all are orthogonal to 1 (hostile)

## Unfriendly graphs

Adjacency matrix has $d$ non-simple eigenspaces
$\underbrace{\lambda_{1}=\lambda_{2}=\cdots=\lambda_{i_{1}}}_{\text {multiplicity } m_{1}+1}<\underbrace{\lambda_{i_{1}+1}=\cdots=\lambda_{i_{1}+i_{2}}}_{\text {multiplicity } m_{2}+1}<\cdots$
$m=m_{1}+m_{2}+\cdots+m_{d}$
Basis vectors of each eigenspace are selected such that either
none of them is orthogonal to 1 (non-hostile); or all are orthogonal to 1 (hostile)
$k=\#$ of hostile eigenspaces

## Unfriendly graphs

Adjacency matrix has $d$ non-simple eigenspaces
$\underbrace{\lambda_{1}=\lambda_{2}=\cdots=\lambda_{i_{1}}}_{\text {multiplicity } m_{1}+1}<\underbrace{\lambda_{i_{1}+1}=\cdots=\lambda_{i_{1}+i_{2}}}_{\text {multiplicity } m_{2}+1}<\cdots$
$m=m_{1}+m_{2}+\cdots+m_{d}$
Basis vectors of each eigenspace are selected such that either
none of them is orthogonal to 1 (non-hostile); or all are orthogonal to 1 (hostile)
$k=\#$ of hostile eigenspaces
Unfriendliness degree: $m+k$

## Matching of unfriendly graphs

## First-order optimality condition:

$$
F_{i j}\left(\lambda_{i}-\lambda_{j}\right)^{2}+v_{j} \gamma_{i}=0 \quad v_{i}=\mathbf{u}_{i}^{\mathrm{T}} \mathbf{1}
$$

Pseudo-stochasticity constraint:

$$
\sum_{j} F_{i j} v_{j}=v_{i}
$$

## Matching of unfriendly graphs

First-order optimality condition:

$$
\left(\begin{array}{ccc}
\left(\lambda_{i}-\lambda_{1}\right)^{2} & & \\
& \ddots & \\
& & \left(\lambda_{i}-\lambda_{n}\right)^{2}
\end{array}\right) \mathbf{f}_{i}+\gamma_{i} \mathbf{v}=\mathbf{0}
$$

Pseudo-stochasticity constraint:

$$
\mathbf{v}^{\mathrm{T}} \mathbf{f}_{i}=v_{i}
$$

for each $i$-th row $\mathbf{f}_{i}=\left(F_{i 1}, \ldots, F_{i n}\right)^{\mathrm{T}}$

## Matching of unfriendly graphs

First-order optimality condition:

$$
\left(\begin{array}{ccc}
\left(\lambda_{i}-\lambda_{1}\right)^{2} & & \\
& \ddots & \\
& & \left(\lambda_{i}-\lambda_{n}\right)^{2}
\end{array}\right) \mathbf{f}_{i}+\gamma_{i} \mathbf{v}=\mathbf{0}
$$

Pseudo-stochasticity constraint:

$$
\mathbf{v}^{\mathrm{T}} \mathbf{f}_{i}=v_{i}
$$

for each $i$-th row $\mathbf{f}_{i}=\left(F_{i 1}, \ldots, F_{\text {in }}\right)^{\mathrm{T}}$
$n$ systems with $n+1$ equations and variables each

## Case I: non-hostile eigenspace

$\mathbf{u}_{i}$ belongs to a non-hostile eigenspace

First-order optimality condition:

$$
\left(\begin{array}{lll}
\left(\lambda_{i}-\lambda_{1}\right)^{2} & & \\
& \ddots & \\
& & \left(\lambda_{i}-\lambda_{n}\right)^{2}
\end{array}\right) \mathbf{f}_{i}+\gamma_{i} \mathbf{v}=\mathbf{0}
$$

Pseudo-stochasticity constraint:

$$
\mathbf{v}^{\mathrm{T}} \mathbf{f}_{i}=v_{i}
$$

## Case I: non-hostile eigenspace

$\mathbf{u}_{i}$ belongs to a non-hostile eigenspace $\Rightarrow v_{i} \neq 0$

First-order optimality condition:

$$
\left(\begin{array}{ccc}
\left(\lambda_{i}-\lambda_{1}\right)^{2} & & \\
& \ddots & \\
& & \left(\lambda_{i}-\lambda_{n}\right)^{2}
\end{array}\right) \mathbf{f}_{i}+\gamma_{i} \mathbf{v}=\mathbf{0}
$$

Pseudo-stochasticity constraint:

$$
\mathbf{v}^{\mathrm{T}} \mathbf{f}_{i}=v_{i}
$$

## Case I: non-hostile eigenspace

$\mathbf{u}_{i}$ belongs to a non-hostile eigenspace $\Rightarrow v_{i} \neq 0$

$$
\Rightarrow \gamma_{i}=0
$$

First-order optimality condition:

$$
\left(\begin{array}{ccc}
\left(\lambda_{i}-\lambda_{1}\right)^{2} & & \\
& \ddots & \\
& & \left(\lambda_{i}-\lambda_{n}\right)^{2}
\end{array}\right) \mathbf{f}_{i}=\mathbf{0}
$$

Pseudo-stochasticity constraint:

$$
\mathbf{v}^{\mathrm{T}} \mathbf{f}_{i}=v_{i}
$$

## Case I: non-hostile eigenspace

$\mathbf{u}_{i}$ belongs to a non-hostile eigenspace $\Rightarrow v_{i} \neq 0$

$$
\Rightarrow \gamma_{i}=0
$$

First-order optimality condition:

$$
\left(\begin{array}{ccc}
\left(\lambda_{i}-\lambda_{1}\right)^{2} & & \\
& \ddots & \\
& & \left(\lambda_{i}-\lambda_{n}\right)^{2}
\end{array}\right) \mathbf{f}_{i}=\mathbf{0}
$$

Pseudo-stochasticity constraint:

$$
\mathbf{v}^{\mathrm{T}} \mathbf{f}_{i}=v_{i}
$$

Rank- $m_{i}$ deficient!

## Case II: hostile eigenspace

$\mathbf{u}_{i}$ belongs to a hostile eigenspace

First-order optimality condition:

$$
\left(\begin{array}{ccc}
\left(\lambda_{i}-\lambda_{1}\right)^{2} & & \\
& \ddots & \\
& & \left(\lambda_{i}-\lambda_{n}\right)^{2}
\end{array}\right) \mathbf{f}_{i}+\gamma_{i} \mathbf{v}=\mathbf{0}
$$

Pseudo-stochasticity constraint:

$$
\mathbf{v}^{\mathrm{T}} \mathbf{f}_{i}=v_{i}
$$

## Case II: hostile eigenspace

$\mathbf{u}_{i}$ belongs to a hostile eigenspace $\Rightarrow v_{i}=0$

First-order optimality condition:

$$
\left(\begin{array}{ccc}
\left(\lambda_{i}-\lambda_{1}\right)^{2} & & \\
& \ddots & \\
& & \left(\lambda_{i}-\lambda_{n}\right)^{2}
\end{array}\right) \mathbf{f}_{i}+\gamma_{i} \mathbf{v}=\mathbf{0}
$$

Pseudo-stochasticity constraint:

$$
\mathbf{v}^{\mathrm{T}} \mathbf{f}_{i}=v_{i}
$$

## Case II: hostile eigenspace

$\mathbf{u}_{i}$ belongs to a hostile eigenspace $\Rightarrow v_{i}=0$
$\Rightarrow \gamma_{i}$ undetermined
First-order optimality condition:

$$
\left(\begin{array}{ccc}
\left(\lambda_{i}-\lambda_{1}\right)^{2} & & \\
& \ddots & \\
& & \left(\lambda_{i}-\lambda_{n}\right)^{2}
\end{array}\right) \mathbf{f}_{i}=-\gamma_{i}\left(\begin{array}{c}
\vdots \\
\mathbf{0} \\
\vdots
\end{array}\right)
$$

Pseudo-stochasticity constraint:

$$
\mathbf{v}^{\mathrm{T}} \mathbf{f}_{i}=0
$$

## Case II: hostile eigenspace

$\mathbf{u}_{i}$ belongs to a hostile eigenspace $\Rightarrow v_{i}=0$
$\Rightarrow \gamma_{i}$ undetermined
First-order optimality condition:

$$
\left(\begin{array}{ccc}
\left(\lambda_{i}-\lambda_{1}\right)^{2} & & \\
& \ddots & \\
& & \left(\lambda_{i}-\lambda_{n}\right)^{2}
\end{array}\right) \mathbf{f}_{i}=-\gamma_{i}\left(\begin{array}{c}
\vdots \\
\mathbf{0} \\
\vdots
\end{array}\right)
$$

Pseudo-stochasticity constraint:

$$
\mathbf{v}^{\mathrm{T}} \mathbf{f}_{i}=0
$$

Rank- $\left(m_{i}+1\right)$ deficient!

## Matching of unfriendly graphs

For an $(m+k)$-unfriendly graph, the system

$$
\begin{aligned}
& F_{i j}\left(\lambda_{i}-\lambda_{j}\right)^{2}+v_{j} \gamma_{i}=0 \\
& \sum_{j} F_{i j} v_{j}=v_{i}
\end{aligned}
$$

is rank- $(m+k)$ deficient!

## Matching of unfriendly graphs

## For an $(m+k)$-unfriendly graph, the system

$$
\begin{aligned}
& F_{i j}\left(\lambda_{i}-\lambda_{j}\right)^{2}+v_{j} \gamma_{i}=0 \\
& \sum_{j} F_{i j} v_{j}=v_{i}
\end{aligned}
$$

is rank- $(m+k)$ deficient!
Solution space is $(m+k)$-dimensional.

## Matching of unfriendly graphs

For an $(m+k)$-unfriendly graph, the system

$$
\begin{aligned}
& F_{i j}\left(\lambda_{i}-\lambda_{j}\right)^{2}+v_{j} \gamma_{i}=0 \\
& \sum_{j} F_{i j} v_{j}=v_{i}
\end{aligned}
$$

is rank- $(m+k)$ deficient!
Solution space is $(m+k)$-dimensional.
Some solutions may belong to Voronoi cells of permutations that are not isomorphisms!

## Matching of unfriendly graphs

For an $(m+k)$-unfriendly graph, the system

$$
\begin{aligned}
& F_{i j}\left(\lambda_{i}-\lambda_{j}\right)^{2}+v_{j} \gamma_{i}=0 \\
& \sum_{j} F_{i j} v_{j}=v_{i}
\end{aligned}
$$

is rank- $(m+k)$ deficient!
Convex relaxation + projection can produce wrong solutions!

## Seeds and attributes

Seeds (known correspondences): collection of $q$ real functions $\mathbf{C}=\left(\mathbf{c}_{1}, \ldots, \mathbf{c}_{q}\right)$ on the vertex set of A with corresponding functions $\mathbf{D}=\left(\mathbf{d}_{1}, \ldots, \mathbf{d}_{q}\right)$ on $\mathbf{B}$.

## Seeds and attributes

Seeds (known correspondences): collection of $q$ real functions $\mathbf{C}=\left(\mathbf{c}_{1}, \ldots, \mathbf{c}_{q}\right)$ on the vertex set of A with corresponding functions $\mathbf{D}=\left(\mathbf{d}_{1}, \ldots, \mathbf{d}_{q}\right)$ on $\mathbf{B}$.

Attributes: $q$-dimensional vector-valued vertex attributes $\mathbf{C}=\left(\mathbf{c}_{1}^{\mathrm{T}}, \ldots, \mathbf{c}_{n}^{\mathrm{T}}\right)^{\mathrm{T}}$.

## Seeds and attributes

Seeds (known correspondences): collection of $q$ real functions $\mathbf{C}=\left(\mathbf{c}_{1}, \ldots, \mathbf{c}_{q}\right)$ on the vertex set of A with corresponding functions $\mathbf{D}=\left(\mathbf{d}_{1}, \ldots, \mathbf{d}_{q}\right)$ on $\mathbf{B}$.

Attributes: $q$-dimensional vector-valued vertex attributes $\mathbf{C}=\left(\mathbf{c}_{1}^{\mathrm{T}}, \ldots, \mathbf{c}_{n}^{\mathrm{T}}\right)^{\mathrm{T}}$.

Covariant with a preferred isomorphism: $\Pi^{*} \mathbf{C}=\mathbf{D}$.

## Seeds and attributes

Seeds (known correspondences): collection of $q$ real functions $\mathbf{C}=\left(\mathbf{c}_{1}, \ldots, \mathbf{c}_{q}\right)$ on the vertex set of A with corresponding functions $\mathbf{D}=\left(\mathbf{d}_{1}, \ldots, \mathbf{d}_{q}\right)$ on $\mathbf{B}$.
Columns of $\mathbf{C}$ and $\Pi^{*} \mathbf{D}$ are corresponding functions (e.g., indicator of vertices).

Attributes: $q$-dimensional vector-valued vertex attributes $\mathbf{C}=\left(\mathbf{c}_{1}^{\mathrm{T}}, \ldots, \mathbf{c}_{n}^{\mathrm{T}}\right)^{\mathrm{T}}$.

Covariant with a preferred isomorphism: $\Pi^{*} \mathbf{C}=\mathbf{D}$.

## Seeds and attributes

Seeds (known correspondences): collection of $q$ real functions $\mathbf{C}=\left(\mathbf{c}_{1}, \ldots, \mathbf{c}_{q}\right)$ on the vertex set of A with corresponding functions $\mathbf{D}=\left(\mathbf{d}_{1}, \ldots, \mathbf{d}_{q}\right)$ on $\mathbf{B}$.
Columns of $\mathbf{C}$ and $\Pi^{*} \mathbf{D}$ are corresponding functions (e.g., indicator of vertices).

Attributes: $q$-dimensional vector-valued vertex attributes $\mathbf{C}=\left(\mathbf{c}_{1}^{\mathrm{T}}, \ldots, \mathbf{c}_{n}^{\mathrm{T}}\right)^{\mathrm{T}}$.
Rows of $\mathbf{C}$ and $\Pi^{*} \mathbf{D}$ are corresponding attributes.
Covariant with a preferred isomorphism: $\Pi^{*} \mathbf{C}=\mathbf{D}$.

## Seeded/attributed graph matching

## Convex Relaxation

$$
\min _{\mathbf{P}}\|\mathbf{P A}-\mathbf{B P}\|_{\mathrm{F}}^{2} \text { s.t. } \mathbf{P} \mathbf{1}=\mathbf{1}
$$

## Seeded/attributed graph matching

## Convex Relaxation of seeded/attributed matching

$$
\min _{\mathbf{P}}\|\mathbf{P A}-\mathbf{B P}\|_{\mathrm{F}}^{2}+\mu\|\mathbf{P C}-\mathbf{D}\|_{\mathrm{F}}^{2} \text { s.t. } \mathbf{P} \mathbf{1}=\mathbf{1}
$$

## Seeded/attributed graph matching

## Convex Relaxation of seeded/attributed matching

$$
\min _{\mathbf{P}}\|\mathbf{P A}-\mathbf{B P}\|_{\mathrm{F}}^{2}+\mu\|\mathbf{P C}-\mathbf{D}\|_{\mathrm{F}}^{2} \text { s.t. } \mathbf{P} 1=\mathbf{1}
$$

penalty on attributes disagreement penalty on seeds correspondence

## Main result

Theorem: Let A and B be isomorphic graphs related by $\Pi^{*}$. Let $\mathbf{C}$ and $\mathbf{D}=\Pi^{*} \mathbf{C}$ be corresponding seeds/attributes, with $\mathbf{D}$ further satisfying for every non-simple eigenspace of $\mathbf{B}$ spanned by $\mathbf{u}_{i}, \ldots, \mathbf{u}_{i+m_{i}}$

- $\mathrm{DD}^{\mathrm{T}} \mathbf{u}_{j} \neq \mathbf{0} \forall j=i, \ldots, i+m_{i}$ if eigenspace is hostile; or
- $\mathbf{D D}^{\mathrm{T}} \mathbf{u}_{j} \neq \mathbf{1} \frac{\mathbf{u}_{i}^{\mathrm{T}} \mathbf{D D}^{\mathrm{T}} \mathbf{u}_{j}}{\mathbf{1}^{\mathrm{T}} \mathbf{u}_{i}} \forall j=i+1, \ldots, i+m_{i}$ otherwise.
Then, $\mathbf{P}^{*}=\boldsymbol{\Pi}^{*}$ is the unique solutuon of the relaxation for every $\mu>0$.


## Sketch of the proof

# Input: two graphs $\mathbf{B}$ and $\mathbf{A}=\boldsymbol{\Pi}^{* T} \mathbf{B} \boldsymbol{\Pi}^{*}$ with seeds/attributes C and $\mathrm{D}=\Pi^{*} \mathrm{C}$ 

## Sketch of the proof

## Input: two graphs $\mathbf{B}$ and $\mathbf{A}=\boldsymbol{\Pi}^{* T} \mathbf{B} \Pi^{*}$ with seeds/attributes C and $\mathrm{D}=\boldsymbol{\Pi}^{*} \mathrm{C}$

Convex quadratic program
$\min _{\mathbf{P}}\|\mathbf{P A}-\mathbf{B P}\|_{\mathbf{F}}^{2}+\mu\|\mathbf{P C}-\mathbf{D}\|_{\mathrm{F}}^{2}$ s.t. $\mathbf{P} \mathbf{1}=\mathbf{1}$
with global minimizer $\mathbf{P}=\boldsymbol{\Pi}^{*}$.

## Sketch of the proof

Input: two graphs $\mathbf{B}$ and $\mathbf{A}=\boldsymbol{\Pi}^{* T} \mathbf{B} \Pi^{*}$ with seeds/attributes C and $\mathrm{D}=\boldsymbol{\Pi}^{*} \mathrm{C}$

Convex quadratic program reparametrized with $\mathbf{Q}=\mathbf{P} \boldsymbol{\Pi}^{* T}$
$\min _{\mathbf{Q}}\|\mathbf{Q B}-\mathbf{B Q}\|_{\mathrm{F}}^{2}+\mu\|\mathbf{Q D}-\mathbf{D}\|_{\mathrm{F}}^{2}$ s.t. $\mathbf{Q} 1=\mathbf{1}$ with global minimizer $\mathbf{Q}=\mathbf{I}$.

## Sketch of the proof

Input: two graphs $\mathbf{B}$ and $\mathbf{A}=\boldsymbol{\Pi}^{* T} \mathbf{B} \boldsymbol{\Pi}^{*}$ with seeds/attributes C and $\mathrm{D}=\boldsymbol{\Pi}^{*} \mathbf{C}$

Convex quadratic program reparametrized with $\mathbf{Q}=\mathbf{P} \Pi^{* T}$
$\min _{\mathbf{Q}}\|\mathbf{Q B}-\mathbf{B Q}\|_{\mathrm{F}}^{2}+\mu\|\mathbf{Q D}-\mathbf{D}\|_{\mathrm{F}}^{2}$ s.t. $\mathbf{Q} 1=\mathbf{1}$
with global minimizer $\mathbf{Q}=\mathbf{I}$.
Show that the minimizer is unique

## Sketch of the proof

## First-order optimality condition: <br> $\mathbf{Q B}^{2}+\mathbf{B}^{2} \mathbf{Q}-2 \mathbf{B Q B}+\mu \mathbf{Q D D}{ }^{\mathrm{T}}-\mu \mathbf{D D}^{\mathrm{T}}+\boldsymbol{\alpha} \mathbf{1}^{\mathrm{T}}=\mathbf{0}$

Pseudo-stochasticity constraint: Q1 = 1

## Sketch of the proof

## First-order optimality condition:

$$
\begin{aligned}
& \mathbf{F} \boldsymbol{\Lambda}^{2}+\boldsymbol{\Lambda}^{2} \mathbf{F}-2 \boldsymbol{\Lambda} \mathbf{F} \boldsymbol{\Lambda}+\mu \mathbf{F G}-\mu \mathbf{G}+\gamma \mathbf{v}^{\mathrm{T}}=\mathbf{0} \\
& \text { with } \mathrm{G}=\mathbf{U}^{\mathrm{T}} \mathbf{D} \mathbf{D}^{\mathrm{T}} \mathbf{U}
\end{aligned}
$$

Pseudo-stochasticity constraint: $\mathbf{F v}=\mathbf{v}$

## Sketch of the proof

First-order optimality condition:

$$
\begin{aligned}
& \mathbf{F} \boldsymbol{\Lambda}^{2}+\boldsymbol{\Lambda}^{2} \mathbf{F}-2 \boldsymbol{\Lambda} \mathbf{F} \boldsymbol{\Lambda}+\mu \mathbf{F G}-\mu \mathbf{G}+\gamma \mathbf{v}^{\mathrm{T}}=\mathbf{0} \\
& \text { with } \mathrm{G}=\mathrm{U}^{\mathrm{T}} \mathbf{D D ^ { \mathrm { T } } \mathrm { U } \succeq 0}
\end{aligned}
$$

Pseudo-stochasticity constraint: $\mathrm{Fv}=\mathrm{v}$

Adding attributes/seeds increases rank

## Main result

Theorem: Let $\mathbf{D}=\mathbf{\Pi}^{*} \mathbf{C}$ satisfying for every non-simple eigenspace $\operatorname{sp}\left\{\mathbf{u}_{i}, \ldots, \mathbf{u}_{i+m_{i}}\right\}$

- $\mathrm{DD}^{\mathrm{T}} \mathbf{u}_{j} \neq 0 \forall j=i, \ldots, i+m_{i}$ if eigenspace is hostile; or
- $\mathbf{D D}^{\mathrm{T}} \mathbf{u}_{j} \neq 1 \frac{\mathbf{u}_{i}^{\mathrm{T}} \mathbf{D D}^{\mathrm{T}} \mathbf{u}_{j}}{\mathbf{1}^{\mathrm{T}} \mathbf{u}_{i}} \forall j=i+1, \ldots, i+m_{i}$ otherwise.
Then, $\mathbf{P}^{*}=\boldsymbol{\Pi}^{*}$ is the unique solutuon of relaxation.


## Main result

Theorem: Let $\mathbf{D}=\mathbf{\Pi}^{*} \mathbf{C}$ satisfying for every non-simple eigenspace $\operatorname{sp}\left\{\mathbf{u}_{i}, \ldots, \mathbf{u}_{i+m_{i}}\right\}$

- $\mathrm{DD}^{\mathrm{T}} \mathbf{u}_{j} \neq 0 \forall j=i, \ldots, i+m_{i}$ if eigenspace is hostile; or
- $\mathrm{DD}^{\mathrm{T}} \mathbf{u}_{j} \neq 1 \frac{\mathbf{u}_{i}^{\mathrm{T}} \mathrm{DD}^{\mathrm{T}} \mathbf{u}_{j}}{\mathbf{1}^{\mathrm{T}} \mathbf{u}_{i}} \forall j=i+1, \ldots, i+m_{i}$ otherwise.
Then, $\mathbf{P}^{*}=\boldsymbol{\Pi}^{*}$ is the unique solutuon of relaxation.
$m+k$ linearly independent seeds are required.


## Experimental validation on 1000 symmetric graphs



## Questions

- Relaxation space: We used $\mathbf{P 1}=1$. Do we need $\mathbf{P} \geq 0$ ? do we need $\mathbf{P}^{\mathrm{T}} \mathbf{1}=\mathbf{1}$ ? Practical consequences?


## Questions

- Relaxation space: We used $\mathbf{P 1}=1$. Do we need $\mathbf{P} \geq 0$ ? do we need $\mathbf{P}^{\mathrm{T}} \mathbf{1}=\mathbf{1}$ ? Practical consequences?
- Better use of geometry: adjacency matrices are, e.g., metric? low dimensional? smooth? bounded curvature?


## Questions

- Relaxation space: We used $\mathbf{P 1}=1$. Do we need $\mathbf{P} \geq 0$ ? do we need $\mathbf{P}^{\mathrm{T}} \mathbf{1}=\mathbf{1}$ ? Practical consequences?
- Better use of geometry: adjacency matrices are, e.g., metric? low dimensional? smooth? bounded curvature?
- Symmetry breaking: add low-rank noise to unfriendly eigenspaces of A to make it friendly. Will the relaxation still work?


## Questions

- Relaxation space: We used $\mathbf{P 1}=1$. Do we need $\mathbf{P} \geq 0$ ? do we need $\mathbf{P}^{\mathrm{T}} \mathbf{1}=\mathbf{1}$ ? Practical consequences?
- Better use of geometry: adjacency matrices are, e.g., metric? low dimensional? smooth? bounded curvature?
- Symmetry breaking: add low-rank noise to unfriendly eigenspaces of A to make it friendly. Will the relaxation still work?
- Finding all isomorphisms (in particular, all symmetries of a graph).

